Normal-mode theory for cylinder arrays

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We study a set of interacting cylinders under the influence of an electromagnetic field in the long-wavelength limit. Cylindrical harmonics are used as basis functions in order to write the electric potential in terms of multipolar moments of the charge distribution in the cylinders. We get a normal-mode expansion where the effects of geometry and material are separated. It is shown that for a row of identical parallel cylinders the electromagnetic modes are distributed symmetrically about the depolarization factor 1/2, each set coupling to different components of the external field. The amplitudes of these symmetric depolarization factors are the same and satisfy proper sum rules.

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I. INTRODUCTION

Interest in the electromagnetic response of arrays of dielectric inclusions has a long history, starting with the seminal work by Lord Rayleigh [1]. Later literature is abundant, including a detailed treatment of square and hexagonal arrays of parallel cylinders using Rayleigh's method [2], and treatment of a pair of parallel cylinders with the method of images [3]. The electromagnetic resonances of arrays of spheres have also been extensively studied [4,5]. A very useful approach to treating these resonances is the spectral representation introduced by Bergman, which separates the effect of geometry from that of the physical properties of the material the cylinders are made of [6,7]. It has been used with advantage in the past for the case of spheres [8,9] as well as cylinders [10].

New techniques for producing nanocylinders [11] or columnar thin films [12] have stimulated further work on cylinder arrays [13]. The usual theoretical approach is to solve Laplace's equation explicitly in special coordinate systems appropriate for the array under consideration. In particular, a technique based on conformal transformations has been applied to small clusters. Another method is to solve the boundary value problem for a single inclusion in an external potential and introduce the coupling among inclusions later through their individual polarizabilities and the local field for each one [14,5]. In this work we take this approach to find the Bergman representation for the case of parallel cylinders. We express the effective polarizability of the system as a normal-mode expansion in terms of resonances and oscillator strengths. Our main result is that the depolarization factors defining these resonances for a row of parallel equal cylinders are symmetrically distributed about the value $\frac{1}{2}$. Only those below $\frac{1}{2}$ are excited when the electric field is in the plane of the cylinder axes, while those above couple to the field component perpendicular to that plane.

In Sec. II we develop the general theory and derive explicit expressions for parallel cylinders, in particular for the case of a row. In Sec. III we treat a pair of parallel identical cylinders and present numerical results for the appropriate depolarization factors and oscillator strengths. Finally, in Sec. IV we present our conclusions.

II. THEORY

We consider an isolated infinite uncharged cylinder of radius a and choose its longitudinal axis as the z axis of coordinates. In terms of radial coordinates (ρ,θ) for the x-y plane, the electric potential it produces $(\rho>a)$ may be written as

$$\Phi(\vec{\rho})=k_c\sum_{m=-\infty}^{\infty}\frac{q_m e^{im\theta}}{|m|\rho^{|m|}},\tag{1}$$

where k_c is a proportionality constant dependent on the chosen system units and q_m is a multipole moment of the charge distribution in the cylinder, defined by the integral

$$q_m=\int d\vec{\rho}\, n(\vec{\rho})\rho^{|m|}e^{-im\theta}.\tag{2}$$

Because there is no net charge in the cylinder, $q_0=0$. We are interested in the dielectric response of the cylinder to the external potential

$$\Phi_{\text{ext}}=k_c V_m e^{im\theta}\rho^m,\tag{3}$$

where m is a positive integer. In terms of this field and the associated multipole excited in the cylinder, we define the m-polar polarizability α_m through

$$q_m=-\alpha_m V_m .\tag{4}$$

The potential created by the charge distribution induced in the cylinder is obtained by solving the boundary value problem with the external potential (3). We then identify the corresponding induced multipolar moment by comparison with relation (1). The resulting expression for the multipolar polarizability is

$$\alpha_m=\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)|m|a^{2|m|}, \tag{5}$$

where ε_1 is the dielectric function of the cylinder and ε_2 is that of the surrounding material.

We next construct a normal-mode expansion for the special case of a system of parallel cylinders. The charge distribution on each depends on the local potential. It is convenient to build this potential by choosing the z axis of coordinates along the axis of the cylinder of interest, say, cylinder j, and then adding up the contributions of all other cylinders. Starting with expression (1) we write the potential close to the origin, produced by a cylinder located at $\vec{\rho}_{j'}$ and having a single multipole moment $q_{m'j'}$,

$$\Phi_{m'j'}=k_c\frac{q_{m'j'}e^{im'\gamma}}{|m'||\vec{\rho}-\vec{\rho}_{j'}|^{|m'|}}, \tag{6}$$

where γ is the angle between vector $\vec{\rho}-\vec{\rho}_{j'}$ and the x axis. As shown in the Appendix, this expression can be written in terms of cylindrical harmonics. Only half of the harmonics are present in the result, depending on the sign of m', and we get

$$\Phi_{m'j'}=k_c\sum_{m>0}^{\infty} V_{mj}^{m'j'}e^{\mp im\theta}\rho^m, \tag{7}$$

with

$$V_{mj}^{m'j'}=\frac{(-1)^{m'}(|m'|+m-1)!}{|m'|!m!}\,\frac{e^{\pm i(|m'|+m)\theta_{j'}}}{\rho_{j'}^{|m'|+m}}q_{m'j'}, \tag{8}$$

where the upper (lower) sign in the exponents corresponds to $m'>0$ $(m'<0)$. Notice that a single multipole moment $q_{m'j'}$ in a cylinder located at $\vec{\rho}_{j'}$ produces near the origin a potential containing all the cylindrical harmonics with the sign of index m opposite to that of m'. Therefore, according to Eq. (4), it induces in cylinder j located at the origin the multipoles

$$q_{mj}=-\alpha_{mj}V_{mj}^{m'j'}, \tag{9}$$

with $mm'<0$. In order to include the effect of all the multipoles in cylinder j' and all the cylinders affecting cylinder j, the previous relation must be added over m' and j'. An additional term V_m representing the cylindrical harmonic component of a potential due to an external source could also be included. Thus, we finally write the following equation showing the coupling among cylinders:

$$q_{mj}=-\alpha_{mj}\left(V_m+\sum_{m'j'}A_{mj}^{m'j'}q_{m'j'}\right), \tag{10}$$

where the coupling coefficients $A_{mj}^{m'j'}$ are defined by

$$A_{mj}^{m'j'}=\begin{cases}0, & \text{if } mm'>0\\ \dfrac{(-1)^{m'}(|m|+|m'|-1)!}{|m|!|m'|!}\,\dfrac{e^{i(m'-m)\theta_{jj'}}}{\rho_{jj'}^{|m|+|m'|}} & \text{if } mm'<0\end{cases} \tag{11}$$

with $\vec{\rho}_{j'}-\vec{\rho}_j=(\rho_{j,j'},\theta_{j,j'})$. The coupling coefficients satisfy the Hermitian relation

$$(A_{mj}^{m'j'})^*=A_{m'j'}^{mj}. \tag{12}$$

In what follows we shall assume for simplicity that $\varepsilon_1=1+4\pi\chi$, $\varepsilon_2=1$. As may be readily verified, the multipolar polarizability α_{mj} given by Eq. (5) may be written in terms of the susceptibility χ of the material the cylinder is made of, as

$$\alpha_{mj}=\frac{2\pi|m|a_j^{2|m|}}{2\pi+\chi^{-1}}. \tag{13}$$

Replacing in Eq. (10) we see that terms depending on the geometry and on the material properties are separated. For a set of parallel cylinders of the same material, the coupled equations can be written as

$$\sum_{\mu'}(\chi^{-1}\delta_{\mu\mu'}+H_{\mu}^{\mu'})x_{\mu'}=f_{\mu}, \tag{14}$$

where μ represents the pair of indices (mj), and

$$H_{mj}^{m'j'}=2\pi(\delta_{mm'}\delta_{jj'}+|mm'|^{1/2}a_j^{|m|}a_{j'}^{|m'|}A_{mj}^{m'j'}), \tag{15}$$

$$f_{mj}=-2\pi|m|^{1/2}a_j^{|m|}V_{mj}, \tag{16}$$

$$x_{mj}=\frac{q_{mj}}{|m|^{1/2}a_j^{|m|}}. \tag{17}$$

The matrix $\mathbf{H}=\{H_{mj}^{m'j'}\}$ has the same Hermitian property as matrix $\mathbf{A}=\{A_{mj}^{m'j'}\}$, a useful property in solving the equations numerically. If the matrix $\mathbf{W}$ diagonalizes $\mathbf{H}$ according to

$$\mathbf{W}^{-1}\mathbf{H}\mathbf{W}=4\pi\mathbf{N}, \tag{18}$$

with $\mathbf{N}=\{n_\mu\delta_{\mu\mu'}\}$, the solutions to the linear equations are given by

$$x_\mu=\sum_{\mu'\mu''}\frac{W_\mu^{\mu'}(W^{-1})_{\mu'}^{\mu''}f_{\mu''}}{\chi^{-1}+4\pi n_{\mu'}}. \tag{19}$$

Here the $n_{\mu'}$ are the usual depolarization factors. The above expression is similar to that for spheres obtained in Refs. [8] and [9]. Taking advantage of the diagonal term in Eq. (15), it is convenient to define a new Hermitian matrix $\mathbf{B}$ through

$$\mathbf{H}=2\pi(\mathbf{I}+\mathbf{B}), \tag{20}$$

$$B_{mj}^{m'j'}=|mm'|^{1/2}a_j^{|m|}a_j^{|m'|}A_{mj}^{m'j'}, \tag{21}$$

where **I** is the unit matrix. Then, if $\{\lambda_\mu\}$ are the eigenvalues of **B** one gets from Eqs. (18) and (20) the relation

$$\lambda_\mu = 2n_\mu - 1. \tag{22}$$

From Eqs. (11) and (21) follows the property

$$B_{mj}^{m'j'} = 0 \quad \text{if} \quad mm' > 0, \tag{23}$$

which can be used in order to reduce to one-half the size of the matrix to be diagonalized. The eigenvalue problem

$$\mathbf{B}Z = \lambda Z \tag{24}$$

can be written in terms of smaller matrices $\mathbf{b}_+$ and $\mathbf{b}_-$, formed by taking elements $B_{mj}^{m'j'}$ with m positive or negative, respectively. In terms of these new matrices, Eq. (24) is split into the two coupled equations

$$\mathbf{b}_+ z_- = \lambda z_+, \tag{25}$$

$$\mathbf{b}_- z_+ = \lambda z_-, \tag{26}$$

where z_+ and z_- are vectors formed by taking components $Z_{m'}$ with m' positive or negative, respectively. The elements of matrix $\mathbf{b}_+$ ($\mathbf{b}_-$) are obtained from the elements of matrix **B**,

$$B_{mj}^{m'j'} = (-1)^{m'} |mm'|^{1/2} a_j^{|m|} a_{j'}^{|m'|} \times \frac{(|m|+|m'|-1)!}{|m|!|m'|!} \frac{e^{i(m'-m)\theta_{jj'}}}{\rho_{jj'}^{|m|+|m'|}} \tag{27}$$

with $m>0$ and $m'<0$ ($m<0$ and $m'>0$).

Up to here our results are valid for an arbitrary distribution of parallel cylinders. A further simplification occurs when we treat a row of parallel cylinders with their axes on a single plane, in which case one obtains $\mathbf{b}_+ = \mathbf{b}_- = \mathbf{b}$, where matrix **b** is real. By taking Eqs. (25) and (26), substitution of one into the other gives two equivalent relations, which we write as

$$\mathbf{b}^2 z_\pm = \lambda^2 z_\pm, \tag{28}$$

where $z_\pm$ stands for z_+ or z_-. By solving

$$\mathbf{b}z = \lambda z \tag{29}$$

we get one-half of the eigenvalues of matrix **B**, the others being just the values $-\lambda$. From this finding and Eq. (22) we get the result that the depolarization factors n_μ form a set of values symmetric about $n=\frac{1}{2}$. Also, to be consistent with Eqs. (25) and (26), we conclude that for the eigenvalue λ of matrix **B**,

$$z_+ = z_-, \tag{30}$$

whereas for eigenvalue $-\lambda$ of the same matrix,

$$z_+ = -z_-. \tag{31}$$

For the case of a row of cylinders, solution (19) may also be given a simpler form. To show this we define a matrix **u** that diagonalizes **b** through

$$\mathbf{u}^{-1}\mathbf{b}\mathbf{u} = \ell, \tag{32}$$

where $\ell_\mu^{\mu'} = \lambda_\mu \delta_{\mu\mu'}$ is now diagonal. In writing the eigenvalue problem for matrix **B** in terms of the smaller matrix **b**,

$$\begin{bmatrix} 0 & \mathbf{b} \\ \mathbf{b} & 0 \end{bmatrix} \begin{bmatrix} z_+ \\ z_- \end{bmatrix} = \lambda \begin{bmatrix} z_+ \\ z_- \end{bmatrix}, \tag{33}$$

it can be shown that **B** can be diagonalized into **L**, according to the relation

$$\mathbf{U}^{-1}\mathbf{B}\mathbf{U} = \mathbf{L}, \tag{34}$$

with **U** and **L** given by

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{u} & \mathbf{u} \\ \mathbf{u} & -\mathbf{u} \end{bmatrix}, \tag{35}$$

$$\mathbf{L} = \begin{bmatrix} \ell & 0 \\ 0 & -\ell \end{bmatrix}. \tag{36}$$

In writing Eq. (14) in terms of matrix **b** we get

$$\begin{bmatrix} (\chi^{-1}+2\pi)\mathbf{I} & 2\pi\mathbf{b} \\ 2\pi\mathbf{b} & (\chi^{-1}+2\pi)\mathbf{I} \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} = \begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \tag{37}$$

which we solve in the form

$$\begin{bmatrix} x_+ \\ x_- \end{bmatrix} = \begin{bmatrix} \mathbf{d}_+ & \mathbf{d}_- \\ \mathbf{d}_- & \mathbf{d}_+ \end{bmatrix} \begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \tag{38}$$

where

$$\mathbf{d}_+ = \tfrac{1}{2}(\mathbf{u}\mathbf{m}_+^{-1}\mathbf{u}^{-1} + \mathbf{u}\mathbf{m}_-^{-1}\mathbf{u}^{-1}), \tag{39}$$

$$\mathbf{d}_- = \tfrac{1}{2}(\mathbf{u}\mathbf{m}_+^{-1}\mathbf{u}^{-1} - \mathbf{u}\mathbf{m}_-^{-1}\mathbf{u}^{-1}), \tag{40}$$

with

$$\mathbf{m}_+ = (\chi^{-1}+2\pi)\mathbf{I} + 2\pi\ell \tag{41}$$

$$\mathbf{m}_- = (\chi^{-1}+2\pi)\mathbf{I} - 2\pi\ell. \tag{42}$$

Because $\mathbf{m}_+$ and $\mathbf{m}_-$ are diagonal, their inverses are also diagonal, and the elements of the inverses are the inverses of the corresponding elements of matrices $\mathbf{m}_+$ and $\mathbf{m}_-$.

In the case of a uniform electric field in the plane of the cylinder axes and perpendicular to them (parallel field geometry), vectors f_+ and f_- representing the external potential are equal and so are the vectors x_+ and x_-. We then have

$$x_- = (\mathbf{d}_+ + \mathbf{d}_-) f_+ \tag{43}$$

with the components of vectors x_- given by

$$x_\mu = \sum_{\mu'} \frac{\mathbf{u}_\mu^{\mu'} \mathbf{u}_1^{\mu'} f_1}{\chi^{-1} + 4\pi n_{\mu'}}, \tag{44}$$

where we have used the orthogonality property of matrix **u**, and

$$f_1 = \pi a E_0, \tag{45}$$

$$n_{\mu'}=\tfrac{1}{2}(1+\lambda_{\mu'}). \tag{46}$$

If, on the other hand, the electric field is in the direction perpendicular to the plane containing the axes of the cylinders (perpendicular field geometry), $f_+=-f_-$ and thus $x_+=-x_-$. Then

$$x_-=(\mathbf{d}_+-\mathbf{d}_-)f_- \tag{47}$$

with the components of the vectors x_- given by

$$x_\mu=\sum_{\mu'}\frac{\mathbf{u}_\mu^{\mu'}\mathbf{u}_1^{\mu'}f_{-1}}{\chi^{-1}+4\pi n_{-\mu'}}, \tag{48}$$

where

$$f_{-1}=i\pi aE_0, \tag{49}$$

$$n_{-\mu'}=\tfrac{1}{2}(1-\lambda_{\mu'}). \tag{50}$$

We note that the eigenvalues λ_μ of matrix $\mathbf{b}$ appear for parallel external field while their negatives are present when the external field is perpendicular. It is easy to verify that expressions (44) and (48) corresponding to parallel and perpendicular geometries are related, as obtained in Ref. [15], a feature deriving from a general property first proved by Keller [16]. Accordingly, the negative of Eq. (48) is obtained from Eq. (44) by substituting ε_1 by $1/\varepsilon_1$. Such a replacement is equivalent to writing $-(\chi^{-1}+4\pi)$ instead of χ^{-1} in Eq. (44). As we shall show in the next section, only depolarization factors greater or smaller than $\frac{1}{2}$ are excited, depending on the direction of the external field, in the case of two parallel cylinders, in accordance with our results (46) and (50).

In general, an arbitrary system of N cylinders described in an M-polar approximation requires arrays of dimension $2MN$. As explained above, however, the property $B_{mj}^{m'j'}=0$ if $mm'>0$ permits a reduction of the dimensionality by a factor of one-half in systems of cylinders with their axes lying on a single plane. In that case, according to Eq. (29), we need just to diagonalize matrix $\mathbf{b}$, of dimension MN.

III. TWO EQUAL CYLINDERS

The simplest array is a system of two ($N=2$) equal and parallel cylinders. It has additional symmetries that permit a further reduction of the dimension of the eigenvalue problem as explained below. By using the property $B_{m,2}^{m',1}=(-1)^{m+m'}B_{m,1}^{m',2}$, the eigenvalue equation for matrix $\mathbf{b}$ as given by Eq. (29) can be written in terms of matrices of dimension M as follows:

$$\begin{bmatrix}0 & \mathbf{g}\\ \overline{\mathbf{g}} & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\lambda\begin{bmatrix}z_1\\ z_2\end{bmatrix}, \tag{51}$$

where $\overline{g}$ is the transpose of g, with the elements of matrix $\mathbf{g}$ given by

$$g_m^{m'}=(-1)^{m'+1}k_m^{m'}, \tag{52}$$

where

$$k_m^{m'}=-\sqrt{mm'}\frac{(m+m'-1)!}{m!m'!}\frac{a^{m+m'}}{\rho^{m+m'}}. \tag{53}$$

Here m and m' are positive integers running from 1 to M. It is easy to verify that the substitution

$$z_{m,2}=(-1)^{m+1}z_{m,1} \tag{54}$$

brings the coupled equations (51) into identical sets of uncoupled eigenvalue equations for vectors z_1 and z_2. Therefore, keeping multipoles up to order M, a single eigenvalue problem of dimension M can be obtained. Thus, the final equation for the normal modes of a pair of identical parallel cylinders excited by a uniform electric field is

$$\mathbf{k}z=\lambda z \tag{55}$$

with $\mathbf{k}=\{k_m^{m'}\}$ given by Eq. (53). The normal-mode expansion for parallel field then results in

$$x_{m,1}=\sum_{m'}\frac{C_m^{m'}f_1}{\chi^{-1}+4\pi n_{m'}^{+}}, \tag{56}$$

where $n_{m'}^{+}$ are given in terms of the eigenvalues λ of the matrix $\mathbf{k}$ according to

$$n_{m'}^{\pm}=\tfrac{1}{2}(1\pm\lambda_{m'}), \tag{57}$$

$f_1=\pi aE_0$, and the coefficients $C_m^{m'}$ are defined by

$$C_m^{m'}=u_m^{m'}u_1^{m'}. \tag{58}$$

Here $\mathbf{u}$ diagonalizes $\mathbf{k}$ and is formed by eigenvectors of the matrix $\mathbf{k}$ ordered in columns. Because $\mathbf{u}$ is orthogonal it can be shown that the coefficients given by relation (58) satisfy the sum rule

$$\sum_{m'}C_m^{m'}=\delta_{m,1}. \tag{59}$$

The corresponding normal-mode expansion for the perpendicular geometry may be written in terms of $C_m^{m'}$ and f_1 as

$$x_{m,1}=-i\sum_{m'}\frac{C_m^{m'}f_1}{\chi^{-1}+4\pi n_{m'}^{-}}, \tag{60}$$

where $n_{m'}^{-}$ is given by Eq. (57).

In solving Eq. (55) numerically for a desired accuracy, a value of M has to be chosen appropriate to a given separation parameter $\sigma=\rho/2a$. A larger value of M is required as σ approaches unity. In order to illustrate such a behavior we show in Fig. 1 the value of M required to achieve convergence in the calculation of n_1, as a function of σ. Convergence is here defined as the value of M for which the first difference smaller than 1% is obtained in the results between successive values. The same convergence data are valid for n_{-1}. We note from the figure that the dipole approximation is sufficient at separations larger than $\sigma=1.90$, whereas the quadrupole is required in the range $1.40\leqslant\sigma<1.90$, the octupole in the range $1.25\leqslant\sigma<1.40$, and so on, with the multi-

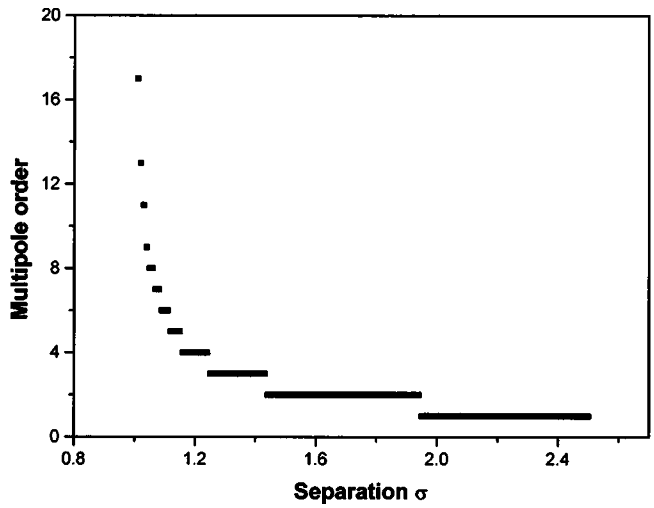

FIG. 1. Multipolar order required for convergence as a function of parameter $\sigma=\rho/2a$ for a system of two equal cylinders.

pole order rapidly increase with decreasing σ. Figure 2 shows the converged depolarization factors n_m corresponding to $m=\pm 1$, as a function of separation. Note that the values are symmetric about 0.50 and approach this value as the separation goes to infinity.

Because optical properties are associated with the dipole moment, we have also studied in detail the coefficients $C_1^{m'}$ and associated depolarization factors $n_m^{\pm}$, for $m=1$, in relation to Eqs. (56) and (60). At large separation, when the dipole approximation is appropriate, we find that only the first term in the normal-mode expansion is important. All the depolarization factors approach the value $\frac{1}{2}$, and the coefficient corresponding to the dipole-dipole mode approaches unity while others exhibit a negligible contribution. At closer separation, however, when a higher order approximation is required, several modes have non-negligible coefficients and the associated depolarization factors are different from the value $\frac{1}{2}$. In Fig. 3 we show the first six coefficients C_1^{1-6} in terms of the associated depolarization factors for a pair of equal cylinders. We have chosen $\sigma=1.10$ for the cases of an external field applied in the parallel [Fig. 3(a)] and perpendicular [Fig. 3(b)] geometries. For the results shown, $M=10$ is enough for convergence, although one should keep in mind that for quantities associated with higher polar orders convergence is slower and requires a larger value of M, as remarked earlier.

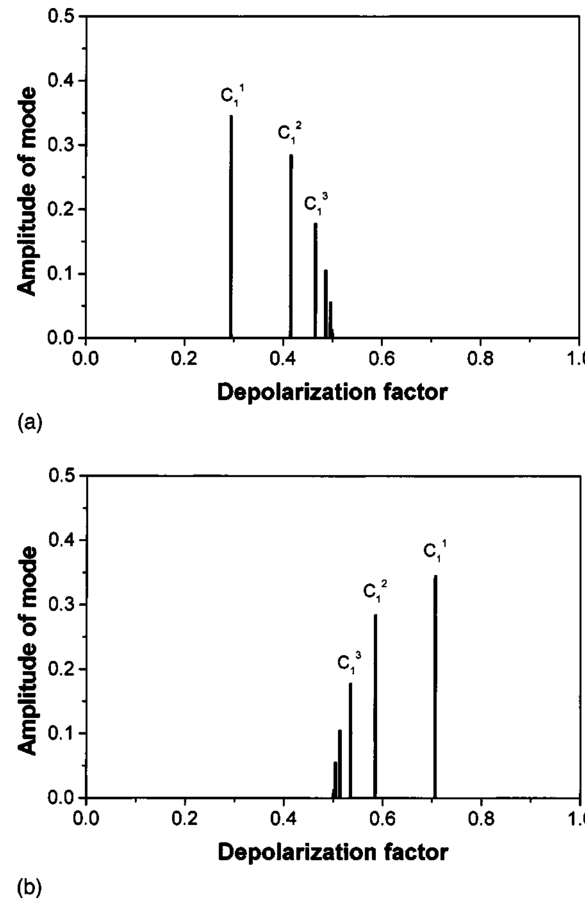

FIG. 3. Amplitudes corresponding to depolarization factors for a pair of equal cylinders with separation parameter $\sigma=1.10$ under uniform electric field. Case (a) is for the parallel field and and case (b) for the perpendicular configuration.

For an isolated cylinder the only mode excited by a perpendicular field is the center mode $n=0.5$. For interacting cylinders, however, the coupling gives rise to the excitation of other modes as well as suggested by Fig. 3. Note the perfect symmetry in size and location about $n=0.5$ of the one case with respect to the other, a result that does not hold for the case of spheres. Owing to the fact that parallel modes are those shifted to the left of the center mode while perpendicular modes are shifted to the right, in optical absorption resonances are redshifted in the former case and blueshifted in the latter. Note also that as the cylinders approach each other not only are the resonances shifted, but also their strength is displaced to higher order modes, making them progressively more important. Although the results shown are for a pair, the qualitative behavior discussed above characterizes other arrays as well, since proximity effects tend to dominate [5].

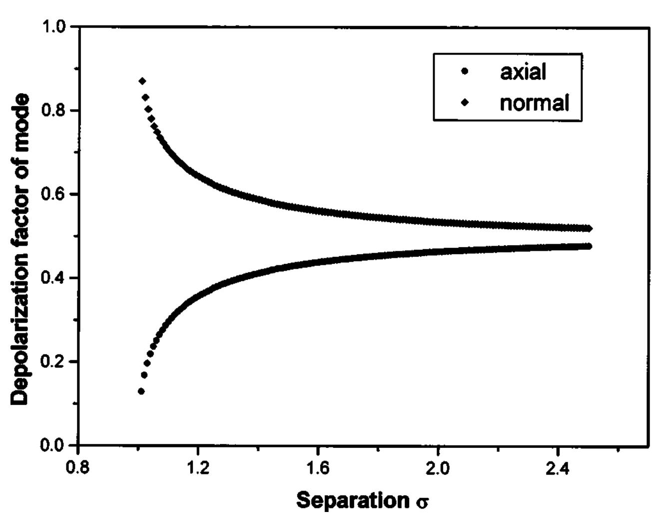

FIG. 2. Depolarization factors corresponding to dipole modes $(m=\pm 1)$ as a function of parameter σ for a pair of equal cylinders.

IV. CONCLUSIONS

We have described a system of interacting dielectric cylinders excited by a uniform external electric field in terms of multipoles of the charge distributions. We have obtained a set of linear equations for the coupled resonances and multipoles, where dielectric properties are separated from the geometry. Our description is similar to a previous one used for spheres, and is an alternative to that proposed recently in Ref. [10]. Our main result for a row of parallel cylinders is that the electromagnetic modes have depolarization factors symmetric about $n=\frac{1}{2}$, according to the relation

$$n_{\pm}=\tfrac{1}{2}(1\pm|\lambda|). \tag{61}$$

This expression is similar to Eq. (3.18) of Ref. [7], obtained by considering just the main term of the interaction between two identical grains. Also, as pointed out in Sec. II, it is consistent with Keller's theorem and provides the appropriate relationship that allows one to get the perpendicular response from the parallel response by simply replacing ε_1 by $1/\varepsilon_1$ and changing the sign of the resulting effective polarizability.

We have also shown that modes with $n<\frac{1}{2}$ are excited by an electric field along a perpendicular line joining the cylinder axes (those with $n>\frac{1}{2}$ have zero amplitude), while those with $n>\frac{1}{2}$ are excited when the field is perpendicular to the plane containing these axes. A similar result is also given for the case of identical grains in Ref. [7], Eq. (3.20), where it is shown that the amplitude corresponding to one of the symmetric eigenvalues is zero. Our work shows that amplitudes corresponding to symmetric eigenvalues depend on the orientation of the external field, one of the amplitudes being zero just for the parallel or the perpendicular geometries. In relation (61) the λ are the eigenvalues of the interaction matrix and depend on geometry only. Furthermore, the amplitudes of the electromagnetic modes are determined just by eigenvectors of the interaction matrix, independent of the direction of the external field, so that amplitudes corresponding to a pair of symmetric depolarization factors are the same. These are exact results.

We have also studied in detail a system of two equal cylinders by solving the linear equations numerically, showing that at edge to edge separation less than about a diameter, multipoles higher than the dipole must be taken into account in the calculations.

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APPENDIX: MATHEMATICAL IDENTITIES

The following identities are required in order to write Eq. (6) in terms of cylindrical harmonics. Assuming $m>0$,

$$\left(\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}\right)^m \ln\frac{1}{|\vec{\rho}-\vec{\rho}_j|}=(-1)^m[2(m-1)]!!\frac{e^{\pm im\alpha}}{|\vec{\rho}-\vec{\rho}_j|^m}, \tag{A1}$$

$$\left(\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}\right)^m \ln\frac{1}{|\vec{\rho}-\vec{\rho}_j|}=2^{m-1}\sum_{k\geqslant m}\frac{(k-1)!}{(k-m)!}\frac{\rho^{k-m}}{\rho_j^k}\times e^{\pm i(-k+m)\theta}e^{\pm ik\theta_j}. \tag{A2}$$

For $m=1$, Eq. (A1) is satisfied by taking the first derivatives of the function $\ln|\vec{\rho}-\vec{\rho}_j|$ directly, while Eq. (A2) requires evaluation of the first derivative of the right hand side of

$$\ln\frac{1}{|\vec{\rho}-\vec{\rho}_j|}=\ln\frac{1}{\rho_j}+\sum_{k=1}^{\infty}\frac{1}{2k}\left(\frac{\rho}{\rho_j}\right)^k(e^{ik(\theta-\theta_j)}+e^{-ik(\theta-\theta_j)}), \tag{A3}$$

an identity found in Ref. [17]. After this is done, the identities follow from mathematical induction.

The exponential function $e^{\pm im\alpha}$ in the first identity can be written as $e^{im\alpha}$ by using a new index m that can now be positive or negative. Comparing Eqs. (A1) and (A2) one then obtains

$$\frac{e^{im\alpha}}{|\vec{\rho}-\vec{\rho}_j|^{|m|}}=\frac{(-1)^m 2^{|m|-1}}{[2(|m|-1)]!!}\sum_{k\geqslant|m|}\frac{(k-1)!}{(k-|m|)!}\frac{\rho^{k-|m|}}{\rho_j^k}\times e^{i\,s(m)(-k+|m|)\theta}e^{i\,s(m)k\theta_j}, \tag{A4}$$

where the function $s(m)$ represents the sign of m. Replacing the index k by $k+|m'|$, Eq. (A4) can be written as

$$\frac{e^{im\alpha}}{|\vec{\rho}-\vec{\rho}_j|^{|m|}}=\frac{(-1)^m}{(|m|-1)!}\sum_{k=0}^{\infty}\frac{(k+|m|-1)!}{k!}\frac{\rho^k}{\rho_j^{k+|m|}}\times e^{\mp ik\theta}e^{\pm i(k+|m|)\theta_j}, \tag{A5}$$

where the upper (lower) sign corresponds to $m>0$ ($m<0$).

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